

Generalized Interval-valued OWA Operators with Interval Weights Derived from Interval-valued Overlap Functions

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Abstract

In this work we extend to the interval-valued setting the notion of an overlap functions and we discuss a method which makes use of interval-valued overlap functions for constructing OWA operators with interval-valued weights. . Some properties of interval-valued overlap functions and the derived interval-valued OWA operators are analysed. We specially focus on the homogeneity and migrativity properties.

Keywords Interval-valued fuzzy sets interval-valued overlap functions Interval-valued overlap OWA operators interval weighted vector migrativity homogeneity

1 Introduction

Interval-valued fuzzy sets [62] have been successfully applied in many different problems. Just to mention some of the most recent ones, interval-valued fuzzy sets have been used in decision making (see, e.g., the works by Khalil and Hassan [36] and Cheng et al. [20]), image processing (see, e.g., the works by Barrenechea et al. [2], Pagola et al. [45] and Melin et al. [39]) or classification (see, e.g., the works by Sanz et al. [52, 53]). They have also been the origin of rich theoretical studies, as, for instance, the works by Bedregal et al. [3, 7], Dimuro et al. [28], Reiser et al. [48] and the recent works by Zywnica et al. [64] and Takác [55].

From the point of view of applications, interval-valued fuzzy sets are a suitable tool to represent uncertain or incomplete information. In particular, the length of the interval-valued membership degree of a given element can be understood as a measure of the lack of certainty of the expert for providing an exact membership value to that element [44]. Interval degrees are also be used to summarize the opinions of several experts. In this case, the left and right interval endpoints can be, for instance, the least and the greatest membership degrees provided by a group of experts. This makes interval-valued fuzzy sets very useful for multiexpert decision making problems, when the experts are asked to express numerically their preferences on several alternatives, as discussed by Bustince et al. [17] (see also the discussions about that in [3, 9, 15, 16]).

Besides, another relevant tool for many different application is that of OWA operators, introduced by Yager [59] and largely used in the literature (see, e.g.: [34, 35, 41]). Its usefulness has led to the consideration of different possible extensions for Atanassov intuitionistic fuzzy sets ([37, 42, 58, 61]) and for interval-valued fuzzy sets ([17, 21, 57, 63]).

In the latter case, however, one of the key problems is how to build and normalize interval-valued weights. In the literature, interval-valued weights are used in several contexts, in order to face the problem of real-world applications in which there are a lot of uncertainty involved and lack of consensus among the modeling experts. Pavlacka [47] presented a review of the existing methods for normalization of interval weights. For example, in the context of multi-criterion decision making, Wang and Li [56] used a hierarchical structure to aggregate local interval weights into global interval weights, by means of a pair of linear programming models to maximize the lower and upper bounds of the aggregated interval value.

However, in the definitions of interval-valued OWA operators found in the literature, the weighted vector is composed, in general, by real numbers. Due to this limitation of the actual models of interval-valued OWA operators, in this paper, we propose the use of interval weights. In order to define these weights we propose the extension of the so-called overlap functions [8, 11, 14, 25, 27] to the interval-valued setting. In this way, the normalization method proposed here makes use of the properties of aggregation functions, and, thus, it is defined in flexible terms.

Then, the objectives of this paper are:

- To introduce the concept of interval-valued overlap functions, and to analyze some of its most relevant properties, such as migrativity and homogeneity;
- To define the normalization of an interval-valued weighted vector by means of a general aggregation function, and to determine which conditions normalized weighted vectors should fulfill;
- To develop a construction method of interval-valued OWA operators based on interval-valued overlap functions, considering interval-valued weights;
- To study the properties of such OWA operators, specially considering the migrativity and homogeneity of interval-valued overlap functions.

This work is organized as follows. In Section 2, we recall some basic concepts that we are going to use along the paper. Next, we define the basic order relations between intervals and, in Section 4, we define the concept of interval-valued overlap function and we study some of its most important properties. In Section 5 we present the concept of normalized weighted vector, and we analyze the definition of interval-valued OWA operators with interval weights. We also study the conditions that the functions used for the definition must fulfill to recover idempotency and other properties. We finish with conclusions and references.

2 Preliminary Concepts

We start, in this section, recalling some well-known concepts which will be necessary for our subsequent developments.

Consider a function $f : [0, 1]^n \rightarrow [0, 1]$. Given $i \in \{1, \dots, n\}$, we say that the component i is necessary if it does not exist a function

$$g : \underbrace{[0, 1] \times \dots \times [0, 1]}_{i-1} \times \underbrace{[0, 1] \times \dots \times [0, 1]}_{n-i} \rightarrow [0, 1]$$

such that

$$f(x_1, \dots, x_n) = g(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n),$$

for any $(x_1, \dots, x_n) \in [0, 1]^n$.

2.1 Aggregation functions

A crucial concept for the present paper is that of aggregation function (see [18]).

Definition 1 A n -ary aggregation function is a mapping $M : [0, 1]^n \rightarrow [0, 1]$ such that

(M1) $M(0, \dots, 0) = 0$ and $M(1, \dots, 1) = 1$;

(M2) M is increasing in each argument: for every $i = 1, \dots, n$, if $x_i \leq y_i$ then $M(x_1, \dots, x_n) \leq M(y_1, \dots, y_n)$.

Several other properties can be required for aggregation functions. In particular, in this work we are interested in the following two ones.

(M3) If $M(x_1, \dots, x_n) = 0$ then there is $i = 1, \dots, n$ such that $x_i = 0$;

(M4) If $M(x_1, \dots, x_n) = 1$ then there is $i = 1, \dots, n$ such that $x_i = 1$.

Among the class of aggregation functions, the so-called OWA operators are a very relevant case. These operators were defined by Yager in [59, 60] in the following way:

Definition 2 Let $w = (w_1, \dots, w_n) \in [0, 1]^n$ be a weighted vector (i.e., $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$). An OWA operator of dimension n associated to the weighted vector w is a function $OWA : [0, 1]^n \rightarrow [0, 1]$ defined by

$$OWA(x_1, \dots, x_n) = \sum_{i=1}^n w_i x_{(i)}$$

where $(.)$ denotes a permutation of $\{1, \dots, n\}$ such that $x_{(1)} \geq x_{(2)} \geq \dots \geq x_{(n)}$.

Another relevant example of aggregation functions is provided by overlap functions [11].

Definition 3 A function $G_O : [0, 1]^2 \rightarrow [0, 1]$ is an overlap function if it satisfies the following conditions:

(GO1) G_O is commutative;

(GO2) $G_O(x, y) = 0$ if and only if $xy = 0$;

(GO3) $G_O(x, y) = 1$ if and only if $xy = 1$;

(GO4) G_O is non-decreasing;

(GO5) G_O is continuous.

Several well-known functions fall into the scope of overlap functions, as, for instance, continuous t-norm without zero divisors, see [8, 25, 27, 26].

It is worth to mention that if an overlap function has a neutral element, then by (GO3) it is necessarily 1. Moreover, associative overlaps always have 1 as neutral element and so they are continuous t-norms without zero divisors [11].

Example 1 Nevertheless, there are overlaps having 1 as neutral element such that they are not associative. For example,

$$G_O(x, y) = \min(x, y) \max(x^2, y^2)$$

or, more generally,

$$G_O(x, y) = \min(x, y) \max(x^p, y^p)$$

for every $p > 0$.

We denote as \mathcal{O} the set of all overlap functions. It follows that $\langle \mathcal{O}, \leq_{\mathcal{O}} \rangle$, where $\leq_{\mathcal{O}}$ is defined for $G_{O_1}, G_{O_2} \in \mathcal{O}$ by

$$G_{O_1} \leq_{\mathcal{O}} G_{O_2} \text{ if and only if } G_{O_1}(x, y) \leq G_{O_2}(x, y)$$

for all $x, y \in [0, 1]$, is a lattice [11, Theorem 3]. In particular, the supremum and infimum of two arbitrary overlap functions G_{O_1} and G_{O_2} are again overlap functions

$$\begin{aligned} G_{O_1} \vee G_{O_2}(x, y) &= \max(G_{O_1}(x, y), G_{O_2}(x, y)) \text{ and} \\ G_{O_1} \wedge G_{O_2}(x, y) &= \min(G_{O_1}(x, y), G_{O_2}(x, y)) \end{aligned} \quad (1)$$

The class of overlap functions is also convex, as was proved in [11]. That is, if we take two overlap functions G_{O_1} and G_{O_2} . then for each $w_1, w_2 \in [0, 1]$ such that $w_1 + w_2 = 1$ we have that their convex sum

$$G_O(x, y) = w_1 G_{O_1}(x, y) + w_2 G_{O_2}(x, y) \quad (2)$$

is also an overlap [11, Corollary 1].

2.2 Interval-valued fuzzy sets

We denote by $L([0, 1]) = \{[a, b] \in [0, 1]^2 : a \leq b\}$ be the set of all closed subintervals of the unit interval $[0, 1]$. In order to simplify the notation, we use the projections of an interval $[a, b] \in L([0, 1])$, given by:

$$\underline{[a, b]} = a \text{ and } \overline{[a, b]} = b. \quad (3)$$

Those intervals X such that $\underline{X} = \overline{X}$ are called degenerate intervals or diagonals elements of $L([0, 1])$. We denote by \mathcal{D} the set of all degenerate intervals.

Recall that an interval-valued fuzzy set (IVFS) \mathcal{A} on a universe U is defined by means of an interval membership function $\mu_{\mathcal{A}} : U \rightarrow L([0, 1])$.

Let's consider now the problem of ordering in $L([0, 1])$. First of all, we have the following partial orders:

Product order: $X \leq_{Pr} Y$ iff $\underline{X} \leq \underline{Y}$ and $\overline{X} \leq \overline{Y}$; and

Inclusion order: $X \subseteq Y$ iff $\underline{Y} \leq \underline{X} \leq \overline{X} \leq \overline{Y}$

$(L([0, 1]), \leq_{Pr})$ is a continuous lattice and so it is a bounded lattice ([32, 54]). In fact the bottom of this lattice is $[0, 0]$, its top is $[1, 1]$, the supremum and infimum of two arbitrary intervals $X, Y \in L([0, 1])$ are the following intervals:

$$\begin{aligned} X \vee Y &= [\max(\underline{X}, \underline{Y}), \max(\overline{X}, \overline{Y})], \text{ and} \\ X \wedge Y &= [\min(\underline{X}, \underline{Y}), \min(\overline{X}, \overline{Y})]. \end{aligned}$$

Notice that, in general, it is possible to define a linear order in $L([0, 1])$ as follows:

Definition 4 [12, 17] *A binary relation \preceq on $L([0, 1])$ is an admissible order if it is a linear order on $L([0, 1])$ refining \leq_{Pr} .*

Admissible orders generated by aggregation functions extend the usual product order on the set of intervals. A deep analysis of this kind of orders can be found in [12].

Some operations on $L([0, 1])$ that are used in this paper are defined, for all $X, Y \in L([0, 1])$, as:

Product: $XY = [\underline{XY}, \overline{XY}]$;

Exponentiation 1: $X^{[k_1, k_2]} = [\underline{X}^{k_2}, \overline{X}^{k_1}]$ for $0 < k_1 \leq k_2$;

Exponentiation 2: $X^{-[k_1, k_2]} = [\overline{X}^{-k_1}, \underline{X}^{-k_2}]$ for any $X \in L([0, 1])$ and $0 < k_1 \leq k_2$;

Complement: $X^c = [1 - \overline{X}, 1 - \underline{X}]$; and

Arctan: $Arctan(X) = [\arctan(\underline{X}), \arctan(\overline{X})]$.

When $k_1 = k_2$ usually we will write X^{k_1} instead of $X^{[k_1, k_2]}$.

We discuss now the relation between interval-valued and real-valued functions.

A function $F : L([0, 1])^n \rightarrow L([0, 1])$ is an interval representation of a function $f : [0, 1]^n \rightarrow [0, 1]$ ([5, 28, 51]), if for any $X_1, \dots, X_n \in L([0, 1])$, $f(x_1, \dots, x_n) \in F(X_1, \dots, X_n)$ when $x_i \in X_i$ for $i = 1, \dots, n$.

Notice that the interval representation of a given function f is not unique, in general. In fact, the function $F : L([0, 1])^n \rightarrow L([0, 1])$ defined as $F(X_1, \dots, X_n) = [0, 1]$, for every $X_1, \dots, X_n \in L([0, 1])$, is an interval representation of any function $f : [0, 1]^n \rightarrow [0, 1]$.

If F is an interval representation of some real function then it is inclusion monotonic, that is, $F(X_1, \dots, X_n) \subseteq F(Y_1, \dots, Y_n)$ whenever $X_i \subseteq Y_i$, for each $i = 1, \dots, n$.

The extension of the notion of an aggregation function to the interval-valued setting can be done in the following way.

Definition 5 Let \leq be an order that extends \leq_{Pr} .¹ An interval-valued aggregation function with respect to \leq is a function $M : L([0, 1])^n \rightarrow L([0, 1])$ such that

1. M is increasing in each argument with respect to \leq : for every $i = 1, \dots, n$, if $X_i \leq Y_i$ then $M(X_1, \dots, X_n) \leq M(Y_1, \dots, Y_n)$;
2. $M([0, 0], \dots, [0, 0]) = [0, 0]$ and $M([1, 1], \dots, [1, 1]) = [1, 1]$.

We finish recalling two notions which are of interest when dealing with overlap functions: migrativity and homogeneity.

The concept of α -migrativity was introduced by Durante *et al.* in [29], see also [40, Problem 1.8(b)] and [30]. Santana *et al.* [50] extended, in a natural way, the notion of migrativity to the interval-valued setting.

An interval-valued function $F : L([0, 1])^2 \rightarrow L([0, 1])$ is migrative if for any $\alpha, X, Y \in L([0, 1])$, we have that $F(\alpha X, Y) = F(X, \alpha Y)$.

The following results are analogous to those discussed in [13], so we do not include a proof.

Lemma 6 A function $F : L([0, 1])^2 \rightarrow L([0, 1])$ is migrative if and only if $F(X, Y) = F([1, 1], XY)$.

Proposition 7 [50, Theorem 1.1] A function $F : L([0, 1])^2 \rightarrow L([0, 1])$ is migrative if and only if the interval function $g_F : L([0, 1]) \rightarrow L([0, 1])$, defined, for all $X \in L([0, 1])$, by

$$g_F(X) = F([1, 1], X) \quad (4)$$

is such that $F(X, Y) = g_F(XY)$, for each $X, Y \in L([0, 1])$.

Corollary 8 If $F : L([0, 1])^2 \rightarrow L([0, 1])$ is migrative, then, for any $X \in L([0, 1])$, it holds that $F([1, 1], X) = F(\sqrt{X}, \sqrt{X})$.

With respect to homogeneity, it can also be extended to the interval-valued setting in the following way. An interval function $F : L([0, 1])^n \rightarrow L([0, 1])$ is homogeneous of order $K = [k_1, k_2]$, with $0 < k_1 \leq k_2$, if, for any $\alpha, X_i \in L([0, 1])$ with $i = 1, \dots, n$, the identity

$$F(\alpha X_1, \dots, \alpha X_n) = \alpha^K F(X_1, \dots, X_n)$$

holds.

¹Observe that \leq does need to be a linear order.

3 Interval-Valued Overlaps

In this section, we introduce the concept of interval-valued overlap functions, which is the key concept of this work.

Definition 9 A function $O : L([0, 1])^2 \rightarrow L([0, 1])$ is an interval-valued overlap function if it satisfies the following conditions:

- (O1).- O is commutative;
- (O2).- $O(X, Y) = [0, 0]$ if and only if $XY = [0, 0]$;
- (O3).- $O(X, Y) = [1, 1]$ if and only if $XY = [1, 1]$;
- (O4).- O is monotonic in the second component, i.e. $O(X, Y) \leq_{Pr} O(X, Z)$ when $Y \leq_{Pr} Z$.
- (O5).- O is Moore continuous [1, 43, 51].

Note that, by (O1) and (O4), interval-valued overlaps also are monotonic in the first component. Observe also that the first four points in our definition are analogous to the first four points in Definition 3. In the last point, however, and in order to have a notion of continuity, we take Moore continuity.

Let \mathfrak{O} be the set of all interval-valued overlap functions. We may define on \mathfrak{O} the binary relation:

$$O_1 \leq_{\mathfrak{O}} O_2 \text{ iff } O_1(X, Y) \leq_{Pr} O_2(X, Y), \text{ for all } X, Y \in L([0, 1])$$

Clearly, $\leq_{\mathfrak{O}}$ is a partial order on \mathfrak{O} . Furthermore, and analogously to the case of real-valued overlap functions, we have the following result.

Proposition 10 $(\mathfrak{O}, \leq_{\mathfrak{O}})$ is an unbounded lattice.

Proof: Let O_1 and O_2 be interval-valued overlap functions. Then the functions $O_1 \vee_{\mathfrak{O}} O_2, O_1 \wedge_{\mathfrak{O}} O_2 : L([0, 1])^2 \rightarrow L([0, 1])$ defined by

$$O_1 \vee_{\mathfrak{O}} O_2(X, Y) = O_1(X, Y) \vee O_2(X, Y)$$

and

$$O_1 \wedge_{\mathfrak{O}} O_2(X, Y) = O_1(X, Y) \wedge O_2(X, Y)$$

are clearly the supremum and infimum of O_1 and O_2 . It is not hard to prove that both are also interval-valued overlap functions.

On the other hand, in order to prove that $(\mathfrak{O}, \leq_{\mathfrak{O}})$ is unbounded it is enough to note that for any interval-valued overlap function O and natural number $n \geq 2$, the functions $O^{\vee}, O^n : L([0, 1])^2 \rightarrow L([0, 1])$, defined by

$$O^{\vee}(X, Y) = O(\sqrt[n]{X}, \sqrt[n]{Y})$$

and

$$O^n(X, Y) = O(X^n, Y^n),$$

respectively, also are interval-valued overlap functions and $O^n <_{\mathfrak{O}} O <_{\mathfrak{O}} O^{\vee}$. Therefore, there is neither a least nor a great interval-valued overlap function. \square

In fact, in the above proposition, if we denote by $O^{\infty} = \lim_{n \rightarrow \infty} O^{\vee}$ and $O^\infty = \lim_{n \rightarrow \infty} O^n$, then it holds that that

$$O^{\infty}(X, Y) = \begin{cases} [0, 0] & \text{if } X \vee Y = [0, 0] \\ [1, 1] & \text{otherwise} \end{cases}$$

and

$$O^\infty(X, Y) = \begin{cases} [1, 1] & \text{if } X \wedge Y = [1, 1] \\ [0, 0] & \text{otherwise.} \end{cases}$$

Observe that these functions are not interval-valued overlap functions, since they do not fulfill the Properties (O2) and (O3), respectively.

The relation between overlap functions and t-norms in the real case is preserved in the interval-valued framework.

Proposition 11 *Let O be an interval-valued overlap function. If O is associative then O is a continuous and positive interval-valued t-norm.*

Proof: Let $g_O : L([0, 1]) \rightarrow L([0, 1])$ be the function $g_O(X) = O(X, [1, 1])$. By continuity of O , g_O is also continuous. Since, $g_O([0, 0]) = [0, 0]$ and $g_O([1, 1]) = [1, 1]$ then, for any $X \in L([0, 1])$, there exists $Y \in L([0, 1])$ such that $g_O(Y) = X$. Therefore, by the associativity property and (O3), we have that

$$\begin{aligned} X &= O(Y, [1, 1]) \\ &= O(Y, O([1, 1], [1, 1])) \\ &= O(O(Y, [1, 1]), [1, 1]) \\ &= O(X, [1, 1]). \end{aligned}$$

So, O has $[1, 1]$ as neutral element. Therefore, since, by hypotheses, O is associative, and by definition of interval-valued overlaps it continuous and positive, then O is a continuous and positive interval-valued t-norm. \square

Observe that any interval-valued overlap function that is also an interval-valued t-norm satisfies the condition $O([1, 1], X) = X$. But there are interval-valued overlap functions, which are not associative (and which are not hence a t-norm), that satisfy this property. For instance, take:

$$O(X, Y) = (Y\sqrt{X}) \wedge (X\sqrt{Y})$$

or

$$O(X, Y) = (X \wedge Y)(X^2 \vee Y^2).$$

3.1 Representable interval-valued overlap functions

In this section, we study the representation of interval-valued overlap functions.

Theorem 12 *Let G_{O_1} and G_{O_2} be overlap functions such that $G_{O_1} \leq_O G_{O_2}$. Then the function $\widetilde{G_{O_1} G_{O_2}} : L([0, 1])^2 \rightarrow L([0, 1])$ defined by*

$$\widetilde{G_{O_1} G_{O_2}}(X, Y) = [G_{O_1}(\underline{X}, \underline{Y}), G_{O_2}(\overline{X}, \overline{Y})] \quad (5)$$

is an interval-valued overlap function.

Proof: It is immediate. \square

Analogously to the notion of t-representability in [22], an interval-valued overlap function O is said to be o -representable if there exist overlap functions G_{O_1} and G_{O_2} such that $O = \widetilde{G_{O_1} G_{O_2}}$. G_{O_1} and G_{O_2} are called representatives of O .

Observe, however, that not every interval-valued overlap function is o -representable, for example

$$O(X, Y) = [\underline{X}\overline{X}\underline{Y}\overline{Y}, \overline{X}\overline{Y}]$$

is clearly not o -representable.

Now we intend to characterize those interval-valued overlap functions which are o -representable. We start with the next definition.

Definition 13 Let $F : L([0, 1])^n \rightarrow L([0, 1])$ be a monotonic function. The left and right projections of F are the functions $\underline{F}, \overline{F} : [0, 1]^n \rightarrow [0, 1]$ defined by

$$\begin{aligned}\underline{F}(x_1, \dots, x_n) &= \underline{F}(\underline{[x_1, x_1]}, \dots, \underline{[x_n, x_n]}) \\ \overline{F}(x_1, \dots, x_n) &= \overline{F}(\overline{[x_1, x_1]}, \dots, \overline{[x_n, x_n]}),\end{aligned}\tag{6}$$

respectively.

Proposition 14 Let $O : L([0, 1])^2 \rightarrow L([0, 1])$ be an interval-valued overlap function. If O is strongly positive (SP), that is, it holds that either $\underline{X} = 0$ or $\underline{Y} = 0$ whereas $O(X, Y) = [0, z]$, for some $z \in (0, 1]$, then \underline{Q} as well as \overline{O} are overlap functions.

Proof: For $i = 1, 3, 4$, since O satisfies the Property (Oi), then, from Equation (6), \underline{Q} clearly satisfies the Property (Oi). Note that \underline{Q} is continuous, since it is a composition of two continuous functions – the interval-valued overlap O and the left projection. So, \underline{Q} , and analogously \overline{O} , are overlap functions. It remains to prove that \underline{Q} satisfies (GO2). In fact, if $\underline{Q}(x, y) = 0$ then one has that $O([x, x], [y, y]) = [0, z]$, for some $z \in [0, 1]$. If $z = 0$ then, by (O2), it holds that either $[x, x] = [0, 0]$ or $[y, y] = [0, 0]$, which implies either $x = 0$ or $y = 0$. If $z > 0$, by (SP), we have that $x = 0$ or $y = 0$. On the other hand, if $x > 0$ and $y > 0$ then it holds that $O([x, x], [y, y]) > [0, 0]$, which implies $\underline{Q}(x, y) > 0$. But, when $\underline{Q}(x, y) = 0$, considering what it was proved above, we have that either $x = 0$ or $y = 0$, which is a contradiction, since $x > 0$ and $y > 0$. Therefore, $\underline{Q}(x, y) > 0$ and hence it satisfies (GO2). \square

Remark 1 It is important to point out that there exist some interval overlap functions O such that \underline{Q} is not an overlap function. For instance, if T_L is the Łukasiewicz's t -norm then $O([a, b], [c, d]) = [T_L(a, c), \min(b, d)]$ is an interval overlap function and, then, one has that $\underline{Q} = T_L$, which is not an overlap function, since it is not positive. This fact shows that the Property (SP) is a necessary condition for the Proposition 14.

Now we can characterize o -representable interval-valued overlap functions.

Theorem 15 An interval-valued overlap function O is o -representable if and only if $O = \widetilde{\underline{Q}\overline{O}}$.

Proof: (\Rightarrow) If O is o -representable then, by Theorem 12, it holds that

$$O = \widetilde{G_{O_1} G_{O_2}},\tag{7}$$

for some overlaps G_{O_1} and G_{O_2} such that $G_{O_1} \leq_O G_{O_2}$. Thus,

$$\begin{aligned}G_{O_1}(x, y) &= \underline{[G_{O_1}(x, y), G_{O_2}(x, y)]} && \text{by Eq. (3)} \\ &= \underline{G_{O_1} G_{O_2}([x, x], [y, y])} && \text{by Eq. (5)} \\ &= \underline{O([x, x], [y, y])} && \text{by Eq. (7)} \\ &= \underline{Q}(x, y) && \text{by Eq. (6)}\end{aligned}$$

Analogously, one shows that $G_{O_2} = \overline{O}$. So, by Eq. (7), it holds that $O = \widetilde{\underline{Q}\overline{O}}$.

(\Leftarrow) It is straightforward, following from Proposition 14. \square

In fact, we can go one step further, thanks to the following Lemma.

Lemma 16 Let $F : L([0, 1])^n \rightarrow L([0, 1])$ be a monotonic function. Then, for any $X_i \in L([0, 1])$, with $i = 1, \dots, n$, it holds that

$$F(X_1, \dots, X_n) = [\underline{F}(\underline{X_1}, \dots, \underline{X_n}), \overline{F}(\overline{X_1}, \dots, \overline{X_n})]$$

if and only if F is inclusion monotonic.

Proof: It is an easy and trivial extension for n-ary interval-valued functions of in [28, Theorem 8]. \square

Theorem 17 *An interval-valued overlap function O is o -representable if and only if it is inclusion monotonic.*

Proof: It is straightforward, following from Lemma 16, Proposition 14, and theorems 12 and 15. \square

In [23], the notion of t -representability was introduced and led to the concepts of pseudo t -representability (denoted by \mathcal{T}_T), generalized pseudo t -representability (denoted by $\mathcal{T}_{T,t}$) and a third type without a particular name, which is denoted by \mathcal{T}'_T . It is clear that whenever we substitute the t -norm T by an overlap O , then \mathcal{T}_O are o -representable interval-valued overlap functions, whereas $\mathcal{T}_{O,t}$ and \mathcal{T}'_O are not. However, in the present paper, we provide a more generic class of “representable” interval-valued overlap functions, which properly contains those three classes.

Definition 18 *An interval-valued overlap O is semi o -representable if there exist overlap functions $G_{O_i} : [0, 1]^2 \rightarrow [0, 1]$, with $i = 1, \dots, 8$, and functions $M_1, M_2 : [0, 1]^4 \rightarrow [0, 1]$, such that for each $X, Y \in L([0, 1])$, it holds that:*

$$O(X, Y) = [M_1(G_{O_1}(\underline{X}, \overline{Y}), G_{O_2}(\overline{X}, \underline{Y}), G_{O_3}(\underline{X}, \underline{Y}), G_{O_4}(\overline{X}, \overline{Y})), M_2(G_{O_5}(\underline{X}, \overline{Y}), G_{O_6}(\overline{X}, \underline{Y}), G_{O_7}(\underline{X}, \underline{Y}), G_{O_8}(\overline{X}, \overline{Y}))] \quad (8)$$

Theorem 19 *Let G_{O_i} , with $i = 1, \dots, 8$, be overlap functions and $M_1, M_2 : [0, 1]^4 \rightarrow [0, 1]$ be aggregation functions such that $G_{O_i} \leq_{\mathcal{D}} G_{O_{i+4}}$, for $i = 1, \dots, 4$, it holds that $M_1 \leq M_2$. Then, for the interval-valued overlap function $O : L([0, 1])^2 \rightarrow L([0, 1])$, defined as in Equation (8), it holds that:*

1. If $G_{O_1} = G_{O_2}$, $G_{O_5} = G_{O_6}$ and M_1 and M_2 are commutative in the two first components then O satisfies (O1);
2. If either M_1 or M_2 satisfies the property (M3) with respect to the fourth component, i.e., $M_1(x_1, x_2, x_3, x_4) = 0$ then $x_4 = 0$, and the same for M_2 . Then O satisfies (O2);
3. If either M_1 or M_2 satisfies the property (M4) with respect to the third component, then O satisfies (O3);
4. If $G_{O_i} \leq_{\mathcal{D}} G_{O_{i+4}}$, for $i = 1, \dots, 4$, then O satisfies (O4);
5. O satisfies (O5) if and only if M_1 and M_2 are continuous.

Proof:

1. We start proving that (O1) holds. Take $X, Y \in L([0, 1])$. Then:

$$\begin{aligned} O(X, Y) &= [M_1(G_{O_1}(\underline{X}, \overline{Y}), G_{O_1}(\overline{X}, \underline{Y}), G_{O_3}(\underline{X}, \underline{Y}), G_{O_4}(\overline{X}, \overline{Y})), \\ &\quad M_2(G_{O_5}(\underline{X}, \overline{Y}), G_{O_5}(\overline{X}, \underline{Y}), G_{O_7}(\underline{X}, \underline{Y}), G_{O_8}(\overline{X}, \overline{Y}))] \\ &= [M_1(G_{O_1}(\overline{Y}, \underline{X}), G_{O_1}(\underline{Y}, \overline{X}), G_{O_3}(\underline{Y}, \underline{X}), G_{O_4}(\overline{Y}, \overline{X})), \\ &\quad M_2(G_{O_5}(\overline{Y}, \underline{X}), G_{O_5}(\underline{Y}, \overline{X}), G_{O_7}(\underline{Y}, \underline{X}), G_{O_8}(\overline{Y}, \overline{X}))] \\ &= [M_1(G_{O_1}(\underline{Y}, \overline{X}), G_{O_1}(\overline{Y}, \underline{X}), G_{O_3}(\underline{Y}, \underline{X}), G_{O_4}(\overline{Y}, \overline{X})), \\ &\quad M_2(G_{O_5}(G_{O_5}(\underline{Y}, \overline{X}), \overline{Y}, \underline{X}), G_{O_7}(\underline{Y}, \underline{X}), G_{O_8}(\overline{Y}, \overline{X}))] \\ &= O(Y, X). \end{aligned}$$

2. Assume that M_1 or M_2 satisfy the property (M3) with respect to the fourth component. If $O(X, Y) = [0, 0]$ then one has that:

$$M_1(G_{O_1}(\underline{X}, \overline{Y}), G_{O_2}(\overline{X}, \underline{Y}), G_{O_3}(\underline{X}, \underline{Y}), G_{O_4}(\overline{X}, \overline{Y})) = 0$$

and

$$M_2(G_{O_5}(\underline{X}, \overline{Y}), G_{O_6}(\overline{X}, \underline{Y}), G_{O_7}(\underline{X}, \underline{Y}), G_{O_8}(\overline{X}, \overline{Y})) = 0$$

Since, as M_1 or M_2 satisfies (M3) with respect to the fourth component, then it follows that either $G_{O_4}(\overline{X}, \overline{Y}) = 0$ or $G_{O_8}(\overline{X}, \overline{Y}) = 0$. In both cases, by (O2), one has that either $\overline{X} = 0$ or $\overline{Y} = 0$, and, therefore, either $X = [0, 0]$ or $Y = [0, 0]$. On the other hand, if $XY = [0, 0]$, then one has that $X = [0, 0]$ or $Y = [0, 0]$. Assume that $X = [0, 0]$. It follows that:

$$\begin{aligned} & O([0, 0], Y) \\ &= [M_1(G_{O_1}(0, \overline{Y}), G_{O_2}(0, \underline{Y}), G_{O_3}(0, \underline{Y}), G_{O_4}(0, \overline{Y})), \\ &\quad M_2(G_{O_5}(0, \overline{Y}), G_{O_6}(0, \underline{Y}), G_{O_7}(0, \underline{Y}), G_{O_8}(0, \overline{Y}))] \\ &\quad \text{by Eq. (8)} \\ &= [M_1(0, 0, 0, 0), M_2(0, 0, 0, 0)] \text{ by (O2)} \\ &= [0, 0] \text{ by (A2)} \end{aligned}$$

If $Y = [0, 0]$ we can make an analogous calculation. Therefore, if $XY = [0, 0]$ then it holds that $O(X, Y) = [0, 0]$. Therefore, the Property (O2) holds.

3. It can be done analogously to item 2;

The items 4. and 5. hold straightforwardly, since the composition of monotonic and continuous functions is monotonic and continuous. \square

Example 2 A trivial example of aggregation functions M_1 and M_2 , and overlap functions G_{O_i} , with $i = 1, \dots, 8$, satisfying the conditions of the Theorem 19 can be obtained taking G_{O_i} such that $G_{O_i} = G_{O_j}$, for each $i, j \in \{1, \dots, 8\}$, and considering the aggregation functions $M_1(x_1, \dots, x_4) = x_4$ and $M_2(x_1, \dots, x_4) = x_3$.

Corollary 20 If there exist functions M_1 , M_2 and G_{O_i} , with $i = 1, \dots, 8$, satisfying the condition of Theorem 19, then the interval-valued overlap function O defined as in Equation (8) is semi o-representable.

Semi o-representability and representability are related, as we discuss next.

Proposition 21 Let O be an interval-valued overlap. If O is o-representable or there exists a t -norm T such that either $O = \mathcal{T}_T$ or $O = \mathcal{T}_{T,t}$, for some $t \in [0, 1]$, or $O = \mathcal{T}'_T$, then O is semi o-representable.

Proof: For the first case, that is, if $O = \widetilde{G'_O G_O}$, for some overlaps functions G'_O and G_O , then it is sufficient to consider $G_{O_3} = G'_O$, $G_{O_8} = G_O$, $M_1(x_1, \dots, x_4) = x_3$ and $M_2(x_1, \dots, x_4) = x_4$. For the second case, that is, if $O = \mathcal{T}_T$, then it is sufficient to consider $O_i = T$, for $i = 3, 5, 6$, $M_1(x_1, \dots, x_4) = x_3$ and $M_2(x_1, \dots, x_4) = \max(x_1, x_2)$. For the third case, that is, if $O = \mathcal{T}_{T,t}$, then it is sufficient consider $G_{O_i} = T$, for $i = 3, 5, 6, 8$, $M_1(x_1, \dots, x_4) = x_3$ and $M_2(x_1, \dots, x_4) = \max(x_1, x_2, T(t, x_4))$. For the fourth case, that is, $O = \mathcal{T}'_T$, then it is sufficient to consider $G_{O_i} = T$, for $i = 1, 2, 8$, $M_1(x_1, \dots, x_4) = \min(x_1, x_2)$ and $M_2(x_1, \dots, x_4) = x_4$. \square

Let $F : L([0, 1]) \rightarrow L([0, 1])$ be defined by $F(X) = m(X) + \frac{1}{2}(X - m(X))$, where $m(X)$ is the middle point of X , that is $m(X) = \frac{\underline{X} + \overline{X}}{2}$. [43] An alternative definition of F is the following:

$$F(X) = \left[\frac{\underline{X} + m(X)}{2}, \frac{\overline{X} + m(X)}{2} \right].$$

The function $O : L([0, 1])^2 \rightarrow L([0, 1])$ defined by

$$O(X, Y) = F(X) \wedge F(Y)$$

is an interval-valued overlap (observe that O is Moore continuous because F and the infimum are Moore continuous [1, 51]). Notice that F is not inclusion monotone [1, 43] and, therefore, by Theorem 17, O is not α -representable. Moreover, by a simple calculation, one may show that

$$O(X, Y) = [\min(a\underline{X} + b\overline{X}, a\underline{Y} + b\overline{Y}), \min(b\underline{X} + a\overline{X}, b\underline{Y} + a\overline{Y})] \quad (9)$$

where $a = \frac{3}{4}$ and $b = \frac{1}{4}$. Thus, if there exists $M_1 : [0, 1]^4 \rightarrow [0, 1]$ and overlap functions G_{O_i} , with $i = 1, \dots, 4$, such that

$$M_1(G_{O_1}(\underline{X}, \overline{Y}), G_{O_2}(\overline{X}, \underline{Y}), G_{O_3}(\underline{X}, \underline{Y}), G_{O_4}(\overline{X}, \overline{Y})) = \min(a\underline{X} + b\overline{X}, a\underline{Y} + b\overline{Y}) \quad (10)$$

then, without loss of generality, we can think that $G_{O_i}(x, y) = x$ in some condition and $G_{O_i}(x, y) = y$ in the other cases, in such a way that, for each X and Y , it holds that $\{G_{O_1}(\underline{X}, \overline{Y}), G_{O_2}(\overline{X}, \underline{Y}), G_{O_3}(\underline{X}, \underline{Y}), G_{O_4}(\overline{X}, \overline{Y})\} = \{\underline{X}, \overline{X}, \underline{Y}, \overline{Y}\}$. So, by (O1), one has that $G_{O_1}(x, y) = x$ if and only if $G_{O_2}(x, y) = y$. Thus, for the particular case where $X = [0, 0]$, then it holds that $G_{O_1}(0, 0.5) = 0$ if and only if $G_{O_2}(0, 0.5) = 0.5$, which is in contradiction with the condition (O2). Therefore, $\underline{O(X, Y)}$ for O defined as in Eq. (9) can not be obtained as in Eq. (10).

3.2 Migrative and Homogeneous Interval-valued Overlap Functions

In this subsection we make a study analogous to [11, Proposition 1].²

Proposition 22 *For an interval-valued function $O : L([0, 1])^2 \rightarrow L([0, 1])$, it holds that:*

1. *If F satisfies the Property (O2) then O does not satisfy the self-duality property (SDP), that is, the equation*

$$F(X, Y) = F(X^c, Y^c)^c \quad (11)$$

fails, for some $X, Y \in L([0, 1])$;

2. *If F is migrative then F satisfies the Property (O1);*
3. *If F is homogeneous of order K then $F([0, 0], [0, 0]) = [0, 0]$;*
4. *If F is homogeneous of order $[1, 1]$ and $F([1, 1], [1, 1]) = [1, 1]$ then F is idempotent;*
5. *If F is migrative and idempotent then F is also homogeneous of order $[1, 1]$;*
6. *If F is migrative and has $[1, 1]$ as neutral element then F is homogeneous of order $[2, 2]$.*

Proof:

²Notice that the item (iii) of Proposition 1 in [11] is wrong. To see that, just consider the constant function $G_O(x, y) = 0$.

1. By the Property (O2), we have that $F([0,0], [1,1]) = [0,0]$. On the other hand, by the definition of complement and the Property (O2), we have that $F([0,0]^c, [1,1]^c)^c = F([1,1], [0,0])^c = [0,0]^c = [1,1]$. Therefore, the Equation (11) fails.
2. If F is migrative then $F(X, Y) = F([1,1]X, Y) = F([1,1], XY) = F([1,1], YX) = F(Y, X)$.
3. It follows that $F([0,0], [0,0]) = [0,0]^K F(X, Y) = [0,0]$.
4. It holds that $F(X, X) = X^{[1,1]} F([1,1], [1,1]) = X[1,1] = X$.
5. By Proposition 7, one has that $F(X, Y) = g_F(XY)$. Since F is idempotent, then $g_F(X) = g_F(\sqrt{X}\sqrt{X}) = F(\sqrt{X}, \sqrt{X}) = \sqrt{X}$. Therefore, it holds that $F(\alpha X, \alpha Y) = g_F(\alpha^2 XY) = \alpha\sqrt{XY} = \alpha g_F(XY) = \alpha F(X, Y)$.
6. By Lemma 6 and because $[1,1]$ is neutral element, one has that $F(X, Y) = F([1,1], XY) = XY$. So, it follows that $F(\alpha X, \alpha Y) = \alpha^2 XY = \alpha^2 F(X, Y)$.

□

Corollary 23 *Let $F : L([0,1])^2 \rightarrow L([0,1])$ be an interval-valued migrative function. F is homogeneous of order $[1,1]$ if and only if F is idempotent.*

Proof: It is straightforward, following from items 4 and 5 of Proposition 22. □

Theorem 24 *$O : L([0,1])^2 \rightarrow L([0,1])$ is an interval-valued migrative overlap function if and only if there exists a monotonic and Moore continuous interval function $g_O : L([0,1]) \rightarrow L([0,1])$ such that, for any $X, Y \in L([0,1])$, it holds that $O(X, Y) = g_O(XY)$, $g_O([0,0]) = [0,0]$, $g_O([1,1]) = [1,1]$ and $g(X) \notin \{[0,0], [1,1]\}$ when $X \notin \{[0,0], [1,1]\}$.*

Proof: (\Rightarrow) By Proposition 7, the function $g_O(X) = O([1,1], X)$ is such that $O(X, Y) = g_O(XY)$, for any $X, Y \in L([0,1])$. Since O is monotonic and Moore continuous, then obviously g_O also is. On the other hand, by the Property (O2), one has that $g_O(X) = [0,0]$ if and only if $O(X, [1,1]) = [0,0]$ if and only if $X = [0,0]$. Analogously, it is possible to prove that $g_O(X) = [1,1]$ if and only if $X = [1,1]$.

(\Leftarrow) By Proposition 7, the function $O(X, Y) = g_O(XY)$ is migrative, and since g_O is increasing and Moore continuous, so is O . The symmetry follows from the commutativity of the product. The Properties (O2) and (O3) are straightforward, following from similar properties of g_O . So, O is a migrative interval-valued overlap function. □

Theorem 25 *Let $O : L([0,1])^2 \rightarrow L([0,1])$ be an interval-valued overlap function. If O is migrative then it is o-representable.*

Proof: Let g_O be the Moore-continuous function of Theorem 24. Since g_O is Moore-continuous, then there exist two continuous functions $g_1, g_2 : [0,1] \rightarrow [0,1]$ such that $g_O(X) = [g_1(\underline{X}), g_2(\overline{X})]$ (see p.52 of [43]). Since, g_O is monotonic then clearly g_1 and g_2 are increasing and, since it holds that $g_O([0,0]) = [0,0]$ and $g_O([1,1]) = [1,1]$, then $g_1(0) = g_2(0) = 0$ and $g_1(1) = g_2(1) = 1$. Therefore, by [11, Theorem 9], the functions $G_{O_1}, G_{O_2} : [0,1]^2 \rightarrow [0,1]$, defined by $G_{O_1}(x, y) = g_1(xy)$ and $G_{O_2}(x, y) = g_2(xy)$, respectively, are migrative overlaps functions. We prove that $O = G_{O_1}G_{O_2}$. It follows that:

$$\begin{aligned} O(X, Y) &= g_O(XY) = g_O([\underline{XY}, \overline{XY}]) = [g_1(\underline{XY}), g_2(\overline{XY})] \\ &= [G_{O_1}(\underline{X}, \underline{Y}), G_{O_2}(\overline{X}, \overline{Y})] = G_{O_1}G_{O_2}(X, Y) \end{aligned}$$

□

Proposition 26 *Let $O : L([0, 1])^2 \rightarrow L([0, 1])$ be an o -representable overlap function. O is homogeneous of order $K = [k_1, k_2]$, with $0 < k_1 \leq k_2$, if and only if their representatives are homogeneous overlaps functions of orders k_2 and k_1 , respectively.*

Proof: Since $O : L([0, 1])^2 \rightarrow L([0, 1])$ is an o -representable interval-valued overlap function then by Theorem 12 and Lemma 16, $O = \underline{O}, \overline{O}$ and, by Proposition 14, \underline{O} is an overlap function.

(\Rightarrow) Consider $\alpha, x, y \in [0, 1]$. It follows that:

$$\begin{aligned} \underline{O}(\alpha x, \alpha y) &= \underline{O}([\alpha x, \alpha x], [\alpha y, \alpha y]) && \text{by Eq. (6)} \\ &= \underline{O}([\alpha, \alpha][x, x], [\alpha, \alpha][y, y]) \\ &= \underline{O}([\alpha, \alpha]^K O([x, x], [y, y])) && \text{by homogeneity} \\ &= \alpha^{k_2} \underline{O}([x, x], [y, y]) \\ &= \alpha^{k_2} \underline{O}(x, y) \end{aligned}$$

Therefore, \underline{O} is an homogeneous overlap function of order k_2 . Analogously, \overline{O} is an homogeneous overlap function of order k_1 .

(\Leftarrow) Consider $\alpha, X, Y \in L([0, 1])$. It follows that:

$$\begin{aligned} \widetilde{\underline{O}, \overline{O}}(\alpha X, \alpha Y) &= [\underline{O}(\alpha X, \alpha Y), \overline{O}(\alpha X, \alpha Y)] && \text{by Eq. (5)} \\ &= [\alpha^{k_2} \underline{O}(X, Y), \alpha^{k_1} \overline{O}(X, Y)] \\ &\quad \text{by homogeneity} \\ &= \alpha^K [\underline{O}(X, Y), \overline{O}(X, Y)] \\ &= \alpha^K \underline{O}, \overline{O}(X, Y) && \text{by Eq. (5)} \end{aligned}$$

□

As in the real case, the following result holds.

Theorem 27 *Consider $K = [k_1, k_2]$, where $0 < k_1 \leq k_2$. The unique interval-valued function $F : L([0, 1])^2 \rightarrow L([0, 1])$ that is migrative, homogeneous of order K and satisfies $F([1, 1], [1, 1]) = [1, 1]$ is*

$$F(X, Y) = (XY)^{\frac{K}{[2, 2]}}.$$

Proof: For any $\alpha, X, Y \in L([0, 1])$, by the homogeneity and Theorem 24, one has that $F(\alpha X, \alpha Y) = \alpha^K F(X, Y) = \alpha^K g_F(XY)$. But, also by Theorem 24, it holds that

$$F(\alpha X, \alpha Y) = g_F(\alpha X \alpha Y).$$

So, it follows that $\alpha^K g_F(XY) = g_F(\alpha^2 XY)$. Thus, taking $X = Y = [1, 1]$, we have that $g_F(\alpha^2) = \alpha^K$ and, so, $g_F(\alpha) = \alpha^{\frac{K}{[2, 2]}}$. □

3.3 Interval-valued overlap functions and t-norms

There are several non-equivalent but related interval-valued extensions of t-norms, as in [5, 24, 28, 31, 33]. Here we consider the definition provided by [24], which is more general than the notion of [5, 33] and consistent with the notion of lattice-valued t-norms as given, for example, in [4, 19, 23, 46].

Lemma 28 *If $O : L([0, 1])^2 \rightarrow L([0, 1])$ is an associative interval-valued overlap function then g_O is idempotent and self-contractive, that is, it holds that $g_O(g_O(X)) = g_O(X)$ and $g_O(X) \subseteq X$, respectively.*

Proof: Remember that $g_O : L([0, 1]) \rightarrow L([0, 1])$ is defined by $g_O(X) = O(X, [1, 1])$. Thus, for any $X \in L([0, 1])$, we have that $g_O(X) = O(X, [1, 1]) = O(X, O([1, 1], [1, 1])) = O(O(X, [1, 1]), [1, 1]) = g_O(g_O(X))$, that is, g_O is idempotent. Now, clearly, since O is monotonic, we have that g_O also is monotonic and so their projection $\underline{g_O}$ and $\overline{g_O}$. Analogously, from Moore continuity of O , we have that g_O is also Moore continuous and so their projection. Finally, it is immediate that $\underline{g_O}(0) = \overline{g_O}(0) = 0$ and $\underline{g_O}(1) = \overline{g_O}(1) = 1$. Then, for any $X \in L([0, 1])$, there exist $x, y \in [0, 1]$ such that $\underline{g_O}(x) = \underline{X}$ and $\overline{g_O}(y) = \overline{X}$. Since $\underline{g_O} \leq \overline{g_O}$ then one has that $x \leq y$. Denote $a = \overline{g_O}(x)$ and $b = \underline{g_O}(y)$. Thus, it holds that $g_O([x, x]) = [\underline{g_O}(x), \overline{g_O}(x)] = [\underline{X}, a]$ and $g_O([y, y]) = [\underline{g_O}(y), \overline{g_O}(y)] = [b, \overline{X}]$. Since g_O is monotonic, then $g_O([x, x]) \leq_{Pr} g_O([y, y])$, that is, $[\underline{X}, a] \leq_{Pr} [b, \overline{X}]$. Thus, one has that $a \leq \overline{X}$ and $\underline{X} \leq b$. Therefore, it follows that $g_O([x, x]) \leq_{Pr} X \leq_{Pr} g_O([y, y])$, and, so, by idempotency and isotonicity, it holds that $[\underline{X}, a] \leq g_O(X) \leq [b, \overline{X}]$. Hence, it follows that $\underline{X} \leq \underline{g_O}(X) \leq \overline{g_O}(X) \leq \overline{X}$, which implies that $g_O(X) \subseteq X$. \square

Notice that the self-contractiveness implies in contractiveness in the sense of [6], and so self-contractiveness is stronger than contractiveness.

Theorem 29 *Let $O : L([0, 1])^2 \rightarrow L([0, 1])$ be an associative interval-valued overlap function such that g_O is either surjective or inclusion monotonic. Then O is an interval-valued t -norm.*

Proof: We prove that $[1, 1]$ is a neutral element of O . If g_O is surjective then there exists $Y \in L([0, 1])$ such that

$$g_O(Y) = X \quad (12)$$

and, therefore,

$$g_O(g_O(Y)) = g_O(X). \quad (13)$$

Thus, by the idempotency of g_O (Lemma 28) and by Eq. (13), we have that $g_O(Y) = g_O(X)$. Therefore, by Eq. (12), one has that $g_O(X) = X$ and, then, $O(X, [1, 1]) = g_O(X) = X$, that is, $[1, 1]$ is a neutral element of O . On the other hand, if g_O is inclusion monotonic, then by Lemma 16, one has that

$$g_O(X) = [\underline{g_O}(X), \overline{g_O}(X)]. \quad (14)$$

Therefore, since $[\underline{X}, \underline{X}] \subseteq X$ and $[\underline{X}, \underline{X}] \leq_{Pr} X$, then, by the inclusion monotonicity and contractiveness (Lemma 28), it follows that

$$g_O([\underline{X}, \underline{X}]) \subseteq g_O(X) \subseteq X \quad (15)$$

and

$$g_O([\underline{X}, \underline{X}]) \leq_{Pr} g_O(X). \quad (16)$$

So, from Eq. (15), it holds that $\underline{X} \leq \underline{g_O}(X)$, and, from Eq. (16), one has that $\underline{g_O}(X) \leq \underline{X}$, that is, $\underline{g_O}(X) = \underline{X}$. Analogously, it is possible to prove that $\overline{g_O}(X) = \overline{X}$. Therefore, from Eq. (14), one has that $O(X, [1, 1]) = g_O(X) = X$, that is, $[1, 1]$ is a neutral element of O . \square

4 Interval-valued OWA operators with interval-valued weighted vectors

In this section we propose a definition that generalizes OWA operators to the interval-valued setting. In most of the cases, this generalization is carried out by considering interval-valued inputs, with, however, pointwise weights. Our definition handles both interval-valued inputs and weights. We start introducing the notion of weighing vector in our setting.

Definition 30 *Let $M : L([0, 1])^n \rightarrow L([0, 1])$ be an interval-valued aggregation function. An n -tuple $(W_1, \dots, W_n) \in L([0, 1])^n$ is said to be an M -weighted vector if and only if*

$$M(W_1, \dots, W_n) = [1, 1]. \quad (17)$$

Note that this definition extends the usual definition of weighted vector in the real-valued case. However, since we consider a general interval-valued aggregation function M for normalizing, we get more flexibility.

Remark 2 Notice that:

1. The vector $([1, 1], \dots, [1, 1])$ is a weighted vector for every interval-valued aggregation function M .
2. Consider the function $M : L([0, 1])^n \rightarrow L([0, 1])$, given by

$$M(X_1, \dots, X_n) = [\max(\underline{X}_1, \dots, \underline{X}_n), \max(\overline{X}_1, \dots, \overline{X}_n)]$$

Then $(W_1, \dots, W_n) \in L([0, 1])^n$ is a M -weighted vector if and only if there exists $i_0 \in \{1, \dots, n\}$ such that $W_{i_0} = [1, 1]$.

3. Consider the function $M : L([0, 1])^n \rightarrow L([0, 1])$ given by

$$M(X_1, \dots, X_n) = \left[\min \left(1, \sum_{i=1}^n \underline{X}_i \right), \min \left(1, \sum_{i=1}^n \overline{X}_i \right) \right]$$

Then $(W_1, \dots, W_n) \in L([0, 1])^n$ is a M -weighted vector if and only if $\sum_{i=1}^n \underline{X}_i \geq 1$.

Our definition of weighing vector allows us to introduce the concept of interval-valued OWA operator considering interval-valued weights.

Definition 31 Let $O : L([0, 1])^2 \rightarrow L([0, 1])$ be an interval-valued overlap function such that $O([1, 1], X) = X$, for every $X \in L([0, 1])$. Let $M : L([0, 1])^n \rightarrow L([0, 1])$ be an interval-valued aggregation function such that for every $X_1, \dots, X_n, Y \in L([0, 1])$, the identity

$$M(O(X_1, Y), \dots, O(X_n, Y)) = O(M(X_1, \dots, X_n), Y) \quad (18)$$

holds. Let $W = (W_1, \dots, W_n) \in L([0, 1])^n$ be a M -weighted vector. Then, an interval-valued OWA operator of dimension n is defined as a function

$$IV-GOWA : L([0, 1])^n \rightarrow L([0, 1]),$$

given by

$$IV-GOWA(X_1, \dots, X_n) = M(O(W_1, X_{(1)}), \dots, O(W_n, X_{(n)})),$$

where $(.)$ denotes a permutation of $\{1, \dots, n\}$ such that $X_{(n)} \leq X_{(n-1)} \leq \dots \leq X_{(1)}$ for an admissible order \leq .

Example 3 Let be $M(X_1, \dots, X_n) = (X_1 \cdot \dots \cdot X_n)^{\frac{1}{n}}$ and $O(X, Y) = XY$. Then we have that

$$\begin{aligned} M(O(X_1, Y), \dots, O(X_n, Y)) &= M(X_1 Y, \dots, X_n Y) \\ &= (X_1 \cdot \dots \cdot X_n)^{\frac{1}{n}} Y, \end{aligned}$$

for every $X_1, \dots, X_n, Y \in L([0, 1])$. Then, we are in the setting of Definition 31. Note that the only possible weighted vector in this setting is $W = ([1, 1], \dots, [1, 1])$.

Example 4 Define

$$M(X_1, \dots, X_n) = \begin{cases} [1, 1] & \text{if } \max(X_1, \dots, X_n) = [1, 1] \\ [0, 0] & \text{otherwise.} \end{cases}$$

Then, for every interval-valued overlap function O and $X_1, \dots, X_n, Y \in L([0, 1])$ it holds that:

$$M(O(X_1, Y), \dots, O(X_n, Y)) = O(M(X_1, \dots, X_n), Y).$$

Note that, in this case, (W_1, \dots, W_n) is a weighted vector if and only if the identity $\max(W_1, \dots, W_n) = [1, 1]$ holds (with respect to the admissible order \leq).

The examples 3 and 4 just present two pairs of interval-valued overlap functions and interval-valued aggregation functions that satisfy the Eq. (18) and, therefore, we can define an interval-valued OWA operator from those pairs. Thus, it would be interesting to have a characterization of such pairs. This fact motivate the following question: “How can we characterize the interval-valued overlap functions that satisfy the Equation (18), for some interval-valued aggregation function?” We answer this question in the following proposition.

Proposition 32 Consider $O(X, Y) = XY$ and let $M : L([0, 1])^n \rightarrow L([0, 1])$ be an interval-valued aggregation function. Then the Identity (18) holds if and only if M is an interval-valued homogeneous function of order $[1, 1]$.³

Proof: Firstly, suppose that M is (interval-valued) homogeneous of order $[1, 1]$. Since $O(X, Y) = XY$, then

$$\begin{aligned} M(O(X_1, Y), \dots, O(X_n, Y)) &= M(X_1 Y, \dots, X_n Y) \\ &= Y M(X_1, \dots, X_n) \\ &= O(M(X_1, \dots, X_n), Y) \end{aligned}$$

Conversely, if the Identity (18) holds, it follows that

$$\begin{aligned} M(\alpha X_1, \dots, \alpha X_n) &= M(O(\alpha, X_1), \dots, O(\alpha, X_n)) \\ &= O(\alpha, M(X_1, \dots, X_n)) \\ &= \alpha M(X_1, \dots, X_n) \end{aligned}$$

Therefore, M is homogeneous of order $[1, 1]$. □

4.1 Some properties of IV-GOWA operators

In the following results, we show how some of the most important properties demanded to real-valued OWA are also fulfilled by IV-GOWA operators.

Proposition 33 Let O and M be as in Definition 31. Then, for any M -weighted vector $W \in L([0, 1])^n$, it follows that:

$$IV-GOWA(X, \dots, X) = X,$$

that is, $IV-GOWA$ is idempotent.

Proof: It follows from the properties demanded to M , O and W . □

Proposition 34 Let M and O be as in Definition 31, and $W \in L([0, 1])^n$ be an M -weighted vector. Then, the function $IV-GOWA$ defined in terms of M , O and W is an interval-valued aggregation function with respect to \leq .

³A similar result was presented in [38] for interval-valued t-norms and t-conorms.

Proof: First of all, note that, by Proposition 33, it follows that

$$IV-GOWA([0, 0], \dots, [0, 0]) = [0, 0]$$

and

$$IV-GOWA([1, 1], \dots, [1, 1]) = [1, 1]$$

Moreover, the monotonicity of $IV-GOWA$ follows from the monotonicity with respect to \leq of both M and O . \square

Proposition 35 Consider $i_0 \in \{1, \dots, n\}$. Then, in the setting of Definition 31, assume that $M([0, 0], \dots, [0, 0], X, [0, 0], \dots, [0, 0]) = X$. Consider the vector W defined by:

$$W_i = \begin{cases} [1, 1] & \text{if } i = i_0 ; \\ [0, 0] & \text{otherwise.} \end{cases}$$

Then it holds that:

1. W is an M -weighted vector.
2. $IV-GOWA(X_1, \dots, X_n) = X_{(i_0)}$, where $X_{(i)}$ denotes the i -th largest interval X_1, \dots, X_n , with respect to an admissible total order.

Proof: It is immediate, following from the Definition 31. \square

Proposition 36 Consider

$$M(X_1, \dots, X_n) = \left[\min \left(1, \sum_{i=1}^n \underline{X}_i \right), \min \left(1, \sum_{i=1}^n \overline{X}_i \right) \right]$$

and $O(X, Y) = XY$. Then it holds that:

1. $W = ([\frac{1}{n}, \frac{1}{n}], \dots, [\frac{1}{n}, \frac{1}{n}])$ is a M -weighted vector.
2. The corresponding $IV-GOWA$ operator is given by:

$$IV-GOWA(X_1, \dots, X_n) = \left[\frac{1}{n} \sum_{i=1}^n \underline{X}_i, \frac{1}{n} \sum_{i=1}^n \overline{X}_i \right]$$

Proof: It is immediate. \square

5 Conclusions and future research

We have presented a study about interval-valued overlaps functions, showing conditions to ensure their representability and discussing the important properties of migrativity and homogeneity. We have also introduced the notion of semi o -representability. We have also discussed a method to build interval-valued OWA operators when considering interval-valued weighting vectors.

As a further work, we will investigate ordinal sums and additive generators of interval-valued overlap functions, in the line of what was done in [25, 27], aiming at practical applications. We also will study how to characterize the homogenous interval-valued aggregation functions.

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